Morris-Thorne wormholes with a cosmological constant

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First, the ideas introduced in the wormhole research field since the work of Morris and Thorne are reviewed, namely, the issues of energy conditions, wormhole construction, stability, time machines and astrophysical signatures. Then, spherically symmetric and static traversable Morris-Thorne wormholes in the presence of a generic cosmological constant Λ are analyzed. A matching of an interior solution to the unique exterior vacuum solution is done using directly the Einstein equations. The structure as well as several physical properties and characteristics of traversable wormholes due to the effects of the cosmological term are studied. Interesting equations appear in the process of matching. For instance, one finds that for asymptotically flat and anti–de Sitter spacetimes the surface tangential pressure $\mathcal{P}$ of the thin shell, at the boundary of the interior and exterior solutions, is always strictly positive, whereas for de Sitter spacetime it can take either sign as one would expect, being negative (tension) for relatively high Λ and high wormhole radius, positive for relatively high mass and small wormhole radius, and zero in between. Finally, some specific solutions with Λ, based on the Morris-Thorne solutions, are provided.

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I. REVIEW AND INTRODUCTION

It has now been 15 years since traversable wormhole theory started in earnest from the work of Morris and Thorne [1]. It was first introduced as a tool for teaching general relativity, as well as an attempt to attract young students into the field, for instance those who had read *Contact*, a novel by Carl Sagan that uses a wormhole to shortcut a large astronomical distance, but it has rapidly spread into several branches. These developments culminated with the publication of the book *Lorentzian Wormholes: From Einstein to Hawking* by Visser [2], where a review on the subject up to 1995, as well as new ideas, are developed and hinted at. It is our intention in this Introduction to do a brief review on the subject of wormholes. The subject has grown substantially, and it is now almost out of control. We will focus on the work developed after Visser's book was published (most of the references prior to its releasing are in it), paying attention to the issues that branched out of [1], such as energy conditions, wormhole construction, stability, time machines, and astrophysical signatures.

A. The beginning

It is true that Wheeler [3], before the work of Morris and Thorne [1], had seen wormholes, such as Reissner-Nordström or Kerr wormholes, as objects of the quantum foam connecting different regions of spacetime and operating at the Planck scale, which were transformed later into Euclidean wormholes by Hawking [4] and others, but these Wheeler wormholes were not traversable, one could not cross them from one side to the other and back, and furthermore they would, in principle, develop some type of singularity [5]. Having been a student of Wheeler, and having further learned through Wheeler's interaction with Zel'dovich on the trace energy condition (which states $\rho \geqslant 3p$, with ρ being the energy density and p the pressure of the fluid on its rest frame) that energy conditions are on shaky ground [6,7], Thorne together with his student Morris [1] understood that wormholes, with two mouths and a throat, might be objects of nature, as stars and black holes are. Indeed, it is a basic fact for the construction of traversable wormholes that the null energy condition, the weakest of the conditions, has to be violated.

B. Energy conditions

The weak energy condition says that the energy density of any system at any point of spacetime for any timelike observer is positive (in the frame of the matter this amounts to $\rho > 0$ and $\rho + p \geqslant 0$), and when the observer moves at the speed of the light it has a well defined limit, called the null energy condition ($\rho + p \geqslant 0$). The weak and null energy con-

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ditions are the weakest of the energy conditions (the null being even weaker than the weak); their violation signals that the other energy conditions are also violated. In Hawking and Ellis' book [8] the weak energy condition is considered a physically reasonable energy condition that at least all classical systems should obey. Afterwards it was found that it could be violated for quantum systems, such as in the Casimir effect and Hawking evaporation (see [9] for a short review). It was further found that for quantum systems in classical gravitational backgrounds the weak or null energy conditions could only be violated in small amounts, and a violation at a given time through the appearance of a negative energy state would be overcompensated for by the appearance of a positive energy state soon after. This idea gave rise to the averaged energy condition [10], and to the quantum inequalities which, being intermediate between the pointwise energy conditions and the averaged energy conditions, limit the magnitude of the negative energy violations and the time for which they are allowed to exist, yielding information on the distribution of the negative energy density in a finite neighborhood [11–13]. It seems that the situation has changed drastically. It has been now shown that even classical systems, such as those built from scalar fields nonminimally coupled to gravity, violate all the energy conditions [14] (see also [15] for other violations of the energy conditions). Thus, gradually the weak and null energy conditions, and with it the other energy conditions, might be losing their status of a kind of law.

C. Wormhole construction: A synthesis

Surely, this has had implications on the construction of wormholes. First, in the original paper [1], Morris and Thorne constructed wormholes by hand, that is, one gives the geometry first, which was chosen as spherically symmetric, and then manufactures the exotic matter accordingly. The engineering work was left to an absurdly advanced civilization, which could manufacture such matter and construct these wormholes. Then, once it was understood that quantum effects should enter in the stress-energy tensor, a self-consistent wormhole solution of semiclassical gravity was found [16], presumably obeying the quantum inequalities. These quantum inequalities when applied to wormhole geometries imply that the exotic matter is confined to an extremely thin band of size only slightly larger than the Planck length, in principle preventing traversability [12]. Finally with the realization that nonminimal scalar fields violate the weak energy condition, a set of self-consistent classical wormholes was found [17]. It is fair to say that, though outside this mainstream, classical wormholes were found by Ellis back in 1973 [18], and related self-consistent solutions were found by Bronnikov in 1973 [19], Kodama in 1978 [20], and Clément in 1981 [21], these papers having been written much before the wormhole boom originated from Morris and Thorne's work [1] (see [22] for a short account of these previous solutions). A self-consistent Ellis wormhole was found again by Harris [23] by solving, through an exotic scalar field, an exercise for students posed in [1].

D. Further wormhole construction

Traversable wormhole theory achieved the end of its first stage after the writing of the monograph on the subject by Visser in 1995 [2]. This monograph is fairly complete on citations, so we refer the reader to it for a bibliographic search up to 1995. We refer here to some developments afterwards, quoting older references when appropriate.

Further wormhole construction in general relativity.

Visser led the way through several works. Indeed, Visser [24] constructed wormholes with polyhedrical symmetry in 1989, generalized a suggestion of Roman for a configuration with two wormholes [1] into a Roman ring [25], started a study on generic dynamical traversable wormhole throats [26] in 1997, found classically consistent solutions with scalar fields [14] in 1999, and also found self-dual solutions [27]. Other authors have also made interesting studies. Before Visser's book we can quote the paper by Frolov and Novikov, where they mix wormhole and black hole physics [28]. After the book, particularly interesting wormholes with toroidal symmetry were found by González-Días [29], wormhole solutions inside cosmic strings were found by Clément [30] and Aros and Zamorano [31], wormholes supported by strings by Schein, Aichelburg and Israel [32], rotating wormholes were found by Teo [33], consistent solutions of the Einstein–Yang-Mills theory in connection with primordial wormhole formation were found in [34], theorems for the impossibility of the existence of wormholes in some Einstein-scalar theories were discussed by Saa [35], wormholes with stress-energy tensors of massless neutrinos and other massless fields were discussed by Krasnikov [36], wormholes made of a crossflow of dust null streams were discussed by Hayward [37] and Gergely [38], and self-consistent charged solutions were found by Bronnikov and Grinyok [39].

Wormhole construction with arbitrarily small violations of the energy conditions.

One of the main areas in wormhole research is to try to avoid as much as possible the violation of the null energy condition. For static wormholes the null energy condition is violated [1,2]. Several attempts have been made somehow to overcome this problem: Morris and Thorne had already tried to minimize the violating region in the original paper [1], Visser [24] found solutions where observers can pass the throat without interacting with the exotic matter, which was pushed to the corners, and Kuhfittig [40] found that the region made of exotic matter can be made arbitrarily small. For dynamic wormholes, the violation of the weak energy condition can be avoided, but the null energy condition, more precisely the averaged null energy condition, is not preserved [26,41,42], although in [43] it has been found that the quantity of violating matter can be made arbitrarily small, a result in line with [40] for static wormholes.

Wormhole construction with a cosmological constant Λ.

Some papers have added a cosmological constant to the wormhole construction. Kim [44] found thin-shell solutions in the spirit of Visser [2], Roman [45] found a wormhole solution inflating in time to test whether one could evade the violation of the energy conditions, Delgaty and Mann [46] looked for new wormhole solutions with Λ, and DeBenedec-

tis and Das [47] found a general class with a cosmological constant. Here, we will further study wormholes in a spacetime with a cosmological constant, as will be detailed below.

Wormhole construction in other theories of gravitation.

In alternative theories to general relativity, wormhole solutions have been worked out. In higher dimensions solutions have been found by Chodos and Detweiler [48], Clément [49], and DeBenedictis and Das [50], in Brans-Dicke theory by Nandi and collaborators [51], in Kaluza-Klein theory by Shen and collaborators [52], in Einstein-Gauss-Bonnet theory by Kar [53], Anchordoqui and Bergliaffa found a wormhole solution in a brane world scenario [54] further examined by Barceló and Visser [55], and Koyama, Hayward and Kim [56] examined a two-dimensional dilatonic theory.

E. Stability

To know the stability of an object against several types of perturbation is always an important issue. Wormholes are not an exception. Not many works though are dedicated to the stability theory of wormholes, although the whole formalism developed for relativistic stars and black holes could be readily used in wormholes. Visser [57], Poisson and Visser [58] and Ishak and Lake [59] studied the stability of wormholes made of thin shells and found, in the parameter space $(\mathcal{P}/\Sigma)\times(\text{throat radius/mass})$, where Σ and $\mathcal{P}$ are, respectively, the surface energy density and surface tangential pressure, those wormholes for which there are stable solutions. For the Ellis drainhole [18], Armendáriz-Picón [60] finds that it is stable against linear perturbations, whereas Shinkai and Hayward [61] find this same class unstable to nonlinear perturbations. Bronnikov and Grinyok [39,62] found that the consistent wormholes of Barceló and Visser [17] are unstable.

F. Wormholes as time machines

An important side effect of wormholes is that they can be converted into time machines, by performing a sufficient delay to the time of one mouth in relation to the other. This can be done either by the special relativistic twin paradox method [63] or by the general relativistic redshift way [64]. The importance of wormholes in the study of time machines is that they provide a noneternal time machine, where closed timelike curves appear to the future of some hypersurface, the chronology horizon (a special case of a Cauchy horizon) which is generated in a compact region in this case. Since time travel to the past is in general unwelcome, it is possible to test whether classical or semiclassical effects will destroy the time machine. It is found that classically it can be easily stabilized [2,63]. Semiclassically, there are calculations that favor the destruction [65,66], leading to chronology protection [66], and others that maintain the machine [25,67]. Other simpler systems that simulate a wormhole, such as Misner spacetime which is a species of two-dimensional wormhole, have been studied more thoroughly, with no conclusive answer. For Misner spacetime the debate still goes on, favoring chronology protection [68], disfavoring it [69], and favoring it again [70]. The upshot is that semiclassical calculations will not settle the issue of chronology protection [71]; one needs a quantum gravity, as has been foreseen sometime before by Thorne [72].

G. Towards a unified view: From stars to wormholes

There is now a growing consensus that wormholes are in the same chain of stars and black holes. For instance, González-Días [29] understood that an enormous pressure on the center ultimately meant a negative energy density to open up the tunnel, DeBenedectis and Das [47] mention that the stress-energy supporting the structure consists of an anisotropic brown dwarf ''star,'' and the wormhole joining one Friedmann-Robertson-Walker universe with Minkowski spacetime or joining two Friedmann-Robertson-Walker universes [26] could be interpreted, after further matchings, as a wormhole joining a collapsing (or expanding) star to Minkowski spacetime or a wormhole joining two dynamical stars, respectively. It has also been recognized, and emphasized by Hayward [73], that wormholes and black holes can be treated in a unified way, the black hole being described by a null outer trapped surface, and the wormhole by a timelike outer trapped surface, this surface being the throat where incoming null rays start to diverge [42,73]. Thus, it seems there is a continuum of objects from stars to wormholes passing through black holes, where stars are made of normal matter, black holes of vacuum, and wormholes of exotic matter. Although not so appealing perhaps, wormholes could be called exotic stars.

H. Astrophysical signatures

Stars are common for everyone to see, black holes also inhabit the universe in billions, so one might tentatively guess that wormholes, formed or constructed one way or another, can also appear in large amounts. If they inhabit the cosmological space, they will produce microlensing effects on point sources at noncosmological distances [74], as well as at cosmological distances; in this case gamma-ray bursts could be the objects microlensed [75,76]. If peculiarly large, wormholes will produce macrolensing effects [77].

I. Aim of this paper

In this paper we extend the Morris-Thorne wormhole solutions [1] by including a cosmological constant Λ. Morris-Thorne wormholes, with $\Lambda=0$, have two asymptotically flat regions. By adding a positive cosmological constant, $\Lambda>0$, the wormholes have two asymptotically de Sitter regions, and by adding a negative cosmological constant, $\Lambda<0$, the wormholes have two asymptotically anti–de Sitter regions. There are a number of reasons to study wormholes with generic Λ that a technologically absurdly advanced civilization might construct. For $\Lambda>0$, we know that an inflationary phase of the ultra-early universe demands it, and moreover, from recent astronomical observations, it seems that we live now in a world with $\Lambda>0$. On the other hand, $\Lambda<0$ is the vacuum state for extended theories of gravitation such as supergravity and superstring theories, and, in addition, even within general relativity, a negative cosmological constant permits solutions of black holes with horizons with topology

different from the usual spherical [78,79] (see [80] for a review), which could be turned into wormhole solutions by adding some exotic matter, although we do not attempt it here.

We follow the spirit of the Morris and Thorne paper [1]. We analyze distributions of matter similar to [1] but now with generic Λ, i.e., we analyze spherically symmetric and static traversable Morris-Thorne wormholes in the presence of a cosmological constant. The more complicated issue of the formalism of junction conditions, that Morris and Thorne so well evaded [1], is here treated through the direct use of the Einstein field equation, and the matter content of the thin-shell separating the wormhole from the exterior spacetime is found. In this way, an equation connecting the radial tension at the mouth with the tangential surface pressure of the thin-shell is derived. The structure as well as several physical properties and characteristics of traversable wormholes due to the effects of the cosmological term are studied. We find that for asymptotically flat and anti–de Sitter spacetimes the surface tangential pressure $\mathcal{P}$ of the thin-shell is always strictly positive, whereas for de Sitter spacetime it can take either sign as one could expect, being negative (tension) for relatively high Λ and high wormhole radius, positive for relatively high mass and small wormhole radius, and zero in between. Finally, some specific solutions with Λ, based on the Morris-Thorne solutions, are provided. In presenting these solutions we dwell mostly on the case $\Lambda=0$, and $\Lambda>0$, and comment briefly on $\Lambda<0$. The plan of the paper is as follows: In Sec. II we present the Einstein field equation for a wormhole metric and perform the junction to an external asymptotically Minkowski, de Sitter, or anti–de Sitter spacetime. In Sec. III we give some wormhole geometries, analogous to [1] having $\Lambda=0$, $\Lambda>0$ and $\Lambda<0$, and study some of their properties. In Sec. IV we conclude.

II. EINSTEIN FIELD EQUATION FOR WORMHOLES WITH A GENERIC COSMOLOGICAL CONSTANT Λ

A. The Einstein field equation with generic Λ: Setting the nomenclature

To set the nomenclature, the Einstein field equation with a cosmological constant is given, in a coordinate basis, by $G_{\mu\nu}+\Lambda g_{\mu\nu}=8\pi G c^{-4}T_{\mu\nu}$, in which $G_{\mu\nu}$ is the Einstein tensor, given by $G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$, $R_{\mu\nu}$ is the Ricci tensor, which is defined as a contraction of the Riemann (or curvature) tensor, $R_{\mu\nu}=R^{\alpha}{}_{\mu\alpha\nu}$, and R is the scalar curvature defined as a contraction of the Ricci tensor, $R=R^{\alpha}_{\alpha}$. The Riemann tensor is a function of the second order derivatives of the metric components $g_{\mu\nu}$. $T_{\mu\nu}$ is the stress-energy tensor of the matter, and Λ the cosmological constant [81].

1. The spacetime metric

We will be interested in the spacetime metric, representing a spherically symmetric and static wormhole, given by [1]

$$ds^2=-e^{2\Phi(r)}c^2dt^2+\frac{dr^2}{1-b(r)/r}+r^2(d\theta^2+\sin^2\theta d\phi^2) \tag{1}$$

where (ct,r,θ,ϕ) are the usual spacetime spherical coordinates, and $\Phi(r)$ and $b(r)$ are arbitrary functions of the radial coordinate r. $\Phi(r)$ is designated the redshift function, for it is related to the gravitational redshift, and $b(r)$ denotes the shape function, because as can be shown by embedding diagrams, it determines the shape of the wormhole [1]. The radial coordinate has a range that increases from a minimum value at $r_{\rm o}$, corresponding to the wormhole throat, to a maximum a corresponding to the mouth. At $r_{\rm o}$ one has to join smoothly this spherical volume to another spherical volume copy with r ranging again from $r_{\rm o}$ to a. In addition, one has then to join each copy to the external spacetime from a to ∞, as will be done.

The mathematical analysis and the physical interpretation will be simplified using a set of orthonormal basis vectors. These may be interpreted as the proper reference frame of a set of observers who remain at rest in the coordinate system (ct,r,θ,ϕ), with (r,θ,ϕ) fixed. Denote the basis vectors in the coordinate system as $\mathbf{e}_t$, $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_\phi$. Then, using the following transformation, $\mathbf{e}_{\hat{\mu}}=\Lambda^{\nu}{}_{\hat{\mu}}\mathbf{e}_\nu$, with

$$\Lambda^{\nu}{}_{\hat{\mu}}=\mathrm{diag}[e^{-\Phi},(1-b/r)^{1/2},r^{-1},(r\sin\theta)^{-1}], \tag{2}$$

where the notation means that the nondiagonal terms of the matrix are zero, one finds

$$\begin{aligned}\mathbf{e}_{\hat{t}}&=e^{-\Phi}\mathbf{e}_t\\ \mathbf{e}_{\hat{r}}&=(1-b/r)^{1/2}\mathbf{e}_r\\ \mathbf{e}_{\hat{\theta}}&=r^{-1}\mathbf{e}_\theta\\ \mathbf{e}_{\hat{\phi}}&=(r\sin\theta)^{-1}\mathbf{e}_\phi.\end{aligned} \tag{3}$$

In this basis the metric components assume their Minkowskian form, given by,

$$g_{\hat{\mu}\hat{\nu}}=\eta_{\hat{\mu}\hat{\nu}}=\mathrm{diag}(-1,1,1,1). \tag{4}$$

In the orthonormal reference frame, the Einstein field equation with a generic cosmological constant is given by

$$G_{\hat{\mu}\hat{\nu}}+\Lambda\,\eta_{\hat{\mu}\hat{\nu}}=\frac{8\pi G}{c^4}T_{\hat{\mu}\hat{\nu}}. \tag{5}$$

The Einstein tensor, given in the orthonormal reference frame by $G_{\hat{\mu}\hat{\nu}}=R_{\hat{\mu}\hat{\nu}}-\frac{1}{2}Rg_{\hat{\mu}\hat{\nu}}$, yields for the metric (1) the following nonzero components:

$$G_{\hat{t}\hat{t}}=\frac{b'}{r^2}, \tag{6}$$

$$G_{\hat{r}\hat{r}}=-\frac{b}{r^3}+2\left(1-\frac{b}{r}\right)\frac{\Phi'}{r}, \tag{7}$$

$$G_{\hat{\theta}\hat{\theta}}=\left(1-\frac{b}{r}\right)\left[\Phi''+(\Phi')^2-\frac{b'r-b}{2r(r-b)}\Phi'-\frac{b'r-b}{2r^2(r-b)}+\frac{\Phi'}{r}\right], \tag{8}$$

$$G_{\hat{\phi}\hat{\phi}}=G_{\hat{\theta}\hat{\theta}}, \tag{9}$$

where a prime denotes a derivative with respect to the radial coordinate r.

2. The stress-energy tensor

The Einstein field equation requires that the Einstein tensor be proportional to the stress-energy tensor. In the orthonormal basis the stress-energy tensor, $T_{\hat{\mu}\hat{\nu}}$, must have an identical algebraic structure as the Einstein tensor components, $G_{\hat{\mu}\hat{\nu}}$, i.e., Eqs. (6)–(9). Therefore, the only nonzero components of $T_{\hat{\mu}\hat{\nu}}$ are $T_{\hat{t}\hat{t}}$, $T_{\hat{r}\hat{r}}$, $T_{\hat{\theta}\hat{\theta}}$, and $T_{\hat{\phi}\hat{\phi}}$. These are given an immediate physical interpretation,

$$T_{\hat{t}\hat{t}}=\rho(r)c^2, \quad T_{\hat{r}\hat{r}}=-\tau(r),$$

$$T_{\hat{\theta}\hat{\theta}}=p(r), \quad T_{\hat{\phi}\hat{\phi}}=p(r), \tag{10}$$

in which $\rho(r)$ is the energy density, $\tau(r)$ is the radial tension, with $\tau(r)=-p_r(r)$, i.e., it is the negative of the radial pressure, and $p(r)$ is the pressure measured in the tangential directions, orthogonal to the radial direction. $T_{\hat{\mu}\hat{\nu}}$ may include surface quantities as we will see.

3. The cosmological constant and the total stress-energy tensor

To obtain a physical interpretation of the cosmological constant, one may write the Einstein field equation in the following manner: $G_{\hat{\mu}\hat{\nu}}=8\pi G c^{-4}(T_{\hat{\mu}\hat{\nu}}+T^{(\mathrm{vac})}_{\hat{\mu}\hat{\nu}})$, in which $T^{(\mathrm{vac})}_{\hat{\mu}\hat{\nu}}=-g_{\hat{\mu}\hat{\nu}}(\Lambda c^4/(8\pi G))$ may be interpreted as the stress-energy tensor associated with the vacuum, and in the orthonormal reference frame is given by

$$T^{(\mathrm{vac})}_{\hat{\mu}\hat{\nu}}=\mathrm{diag}[\Lambda c^4/(8\pi G),-\Lambda c^4/(8\pi G),-\Lambda c^4/(8\pi G),-\Lambda c^4/(8\pi G)]. \tag{11}$$

We see it is thus possible to adopt the viewpoint that the cosmological term is an integral part of the stress-energy tensor, being considered as a fluid. Accordingly, we can define the total stress-energy tensor, $\overline{T}_{\hat{\mu}\hat{\nu}}$, as

$$\overline{T}_{\hat{\mu}\hat{\nu}}=T_{\hat{\mu}\hat{\nu}}+T^{(\mathrm{vac})}_{\hat{\mu}\hat{\nu}} \tag{12}$$

such that $G_{\hat{\mu}\hat{\nu}}=8\pi G c^{-4}\overline{T}_{\hat{\mu}\hat{\nu}}$. Thus, the components of the total stress-energy tensor of the wormhole, $\overline{\rho}(r)$, $\overline{\tau}(r)$ and $\overline{p}(r)$, are given by the following functions:

$$\overline{\rho}(r)=\rho(r)+\frac{c^2}{8\pi G}\Lambda, \tag{13}$$

$$\overline{\tau}(r)=\tau(r)+\frac{c^4}{8\pi G}\Lambda, \tag{14}$$

$$\overline{p}(r)=p(r)-\frac{c^4}{8\pi G}\Lambda. \tag{15}$$

This viewpoint may be interesting to adopt in some cases.

4. The Einstein equations

We are interested in matching the interior solution, whose metric is given by Eq. (1), to an exterior vacuum solution, which will be considered below. Using Eq. (5) and equating Eqs. (6)–(8), with Eq. (10), we obtain the following set of equations:

$$\rho(r)=\frac{c^2}{8\pi G}\left(\frac{b'}{r^2}-\Lambda\right), \tag{16}$$

$$\tau(r)=\frac{c^4}{8\pi G}\left[\frac{b}{r^3}-2\left(1-\frac{b}{r}\right)\frac{\Phi'}{r}-\Lambda\right], \tag{17}$$

$$p(r)=\frac{c^4}{8\pi G}\left\{\left(1-\frac{b}{r}\right)\left[\Phi''+(\Phi')^2-\frac{b'r-b}{2r^2(1-b/r)}\Phi'-\frac{b'r-b}{2r^3(1-b/r)}+\frac{\Phi'}{r}\right]+\Lambda\right\}. \tag{18}$$

By taking the derivative of Eq. (17) with respect to the radial coordinate r, and eliminating b' and Φ'', given in Eq. (16) and Eq. (18), respectively, we obtain the following equation:

$$\tau'=(\rho c^2-\tau)\Phi'-\frac{2}{r}(p+\tau). \tag{19}$$

Equation (19) is the relativistic Euler equation, or the hydrostatic equation for equilibrium for the material threading the wormhole, and can also be obtained using the conservation of the stress-energy tensor, $T^{\hat{\mu}\hat{\nu}}{}_{;\hat{\nu}}=0$, putting $\hat{\mu}=r$. The conservation of the stress-energy tensor, in turn, can be deduced from the Bianchi identities, which are equivalent to $G^{\hat{\mu}\hat{\nu}}{}_{;\hat{\nu}}=0$.

5. Method for solving the Einstein equations

The conventional approach to solving the Einstein equations would be to assume a specific and plausible type of matter or fields for the source of the stress-energy tensor. One would then derive equations of state for the radial tension and the tangential pressure, as functions of the energy density. These equations of state, together with the three field equations, would provide the geometry of the spacetime given in terms of the metric, $g_{\mu\nu}$, as we would have five equations for five unknown functions, i.e., $b(r)$, $\Phi(r)$, ρ, τ and p. Morris and Thorne's approach [1], which will be followed in this paper, differs as they first fixed a convenient geometry for a wormhole solution and then derived the matter distribution for the respective solution (see [47] for a careful analysis of the various approaches).

B. Construction of a wormhole with generic Λ. I: General comments

1. The mathematics of embedding

We can use embedding diagrams to represent a wormhole and extract some useful information for the choice of the shape function, $b(r)$, which will be used in the specific solutions considered below. Due to the spherically symmetric nature of the problem, one may consider an equatorial slice, $\theta=\pi/2$, without loss of generality. The respective line element, considering a fixed moment of time, $t=$const, is given by

$$ds^2=\frac{dr^2}{1-b(r)/r}+r^2d\phi^2. \tag{20}$$

To visualize this slice, one embeds this metric into three-dimensional Euclidean space, whose metric can be written in cylindrical coordinates, (r,ϕ,z), as

$$ds^2=dz^2+dr^2+r^2d\phi^2. \tag{21}$$

Now, in the three-dimensional Euclidean space the embedded surface has equation $z=z(r)$, and thus the metric of the surface can be written as,

$$ds^2=\left[1+\left(\frac{dz}{dr}\right)^2\right]dr^2+r^2d\phi^2. \tag{22}$$

Comparing Eq. (22) with Eq. (20) we have the equation for the embedding surface, given by

$$\frac{dz}{dr}=\pm\left(\frac{r}{b(r)}-1\right)^{-1/2}. \tag{23}$$

To be a solution of a wormhole, the geometry has a minimum radius, $r=b(r)=r_0$, denoted as the throat, at which the embedded surface is vertical, i.e., $dz/dr\to\infty$. Outside the wormhole, far from the mouth, space can be asymptotically flat, de Sitter, or anti–de Sitter.

One can define the proper radial distance for the upper part of the wormhole $z>0$ as

$$l(r)=\int_{r_0}^{r}\frac{dr}{[1-b(r)/r]^{1/2}}, \tag{24}$$

and for the lower part $z<0$ as

$$l(r)=-\int_{r_0}^{r}\frac{dr}{[1-b(r)/r]^{1/2}}. \tag{25}$$

The maximum upper limit of integration is a, the radius of the wormhole mouth. One can also verify that the shape function, $b(r)$, is positive in order that $\sqrt{r/b-1}$ is real. This fact will play an important role for constructing specific solutions.

2. Exotic matter

Following [1] closely we will see that the wormhole needs exotic matter, i.e., matter that does not obey the null energy condition, and thus does not obey the weak or any other energy condition. The null energy condition applied to the matter considered in Eqs. (13), (14) is $\overline{\rho}c^2-\overline{\tau}>0$. Thus a good way to define exoticity is through the parameter ξ defined as [1] $\xi=(\overline{\tau}-\overline{\rho}c^2)/|\overline{\rho}c^2|$. This parameter ξ is dimensionless, and when positive signals exotic matter. Using Eqs. (16), (17) one finds

$$\xi=\frac{\overline{\tau}-\overline{\rho}c^2}{|\overline{\rho}c^2|}=\frac{b/r-b'-2r(1-b/r)\Phi'}{|b'-\Lambda r^2|}. \tag{26}$$

To be a solution of a wormhole, one needs to impose that the throat flares out. Mathematically, this flaring-out condition entails that the inverse of the embedding function $r(z)$ must satisfy $d^2r/dz^2>0$ near the throat r_0. Differentiating $dr/dz=\pm[r/b(r)-1]^{1/2}$ with respect to z, we have $d^2r/dz^2=(b-b'r)/2b^2>0$. Combining this with Eq. (26), the exoticity function takes the form

$$\xi=\frac{2b^2}{r|b'-\Lambda r^2|}\frac{d^2r}{dz^2}-2r\left(1-\frac{b}{r}\right)\frac{\Phi'}{|b'-\Lambda r^2|}. \tag{27}$$

Considering the finite character of ρ, and therefore of b', and the fact that $(1-b/r)\Phi'\to0$ at the throat, we have the following relationship:

$$\xi(r_0)=\frac{\overline{\tau}_0-\overline{\rho}_0c^2}{|\overline{\rho}_0c^2|}>0. \tag{28}$$

Thus matter at the throat is exotic (see [1,2,26] for a detailed discussion).

C. Construction of a wormhole with generic Λ. II: Interior and exterior solutions, and junction conditions

We will distinguish the interior cosmological constant, $\Lambda_{\rm int}$, from the exterior one, $\Lambda_{\rm ext}$. Equations (16)–(18) demonstrate that once the geometry is fixed by the redshift function $\Phi(r)$, and the shape function $b(r)$, the inclusion of the cosmological constant $\Lambda_{\rm int}$ will shift the respective values of

$\rho(r)$, $\tau(r)$ and $p(r)$ and might help in minimizing the violation of the energy conditions.

1. Interior solution of the Einstein equations with generic $\Lambda_{\rm int}$

To find an interior solution of the Einstein equations with generic Λ, we combine the analysis developed in the previous sections, taking into account the notation that $\Lambda_{\rm int}$ represents the cosmological constant associated with the interior solution. The respective Einstein equations provide the following relationships:

$$\rho(r)=\frac{c^2}{8\pi G}\left(\frac{b'}{r^2}-\Lambda_{\rm int}\right), \tag{29}$$

$$\tau(r)=\frac{c^4}{8\pi G}\left[\frac{b}{r^3}-2\left(1-\frac{b}{r}\right)\frac{\Phi'}{r}-\Lambda_{\rm int}\right], \tag{30}$$

$$p(r)=\frac{c^4}{8\pi G}\left\{\left(1-\frac{b}{r}\right)\left[\Phi''+(\Phi')^2-\frac{b'r-b}{2r^2(1-b/r)}\Phi'-\frac{b'r-b}{2r^3(1-b/r)}+\frac{\Phi'}{r}\right]+\Lambda_{\rm int}\right\}. \tag{31}$$

The metric quantities should carry a subscript $_{\rm int}$, but we do not include it in order to not overload the notation. It is of interest to find an expression for the radial tension at the throat. From Eq. (30) one finds that at the throat $[b(r_{\rm o})=r_{\rm o}]$ the tension is

$$\tau(r_{\rm o})=\frac{c^4}{8\pi G}\left(\frac{1}{r_{\rm o}^2}-\Lambda_{\rm int}\right). \tag{32}$$

Thus the radial tension at the throat is positive for wormholes whose structure yields $\Lambda_{\rm int}<1/r_{\rm o}^2$; this includes wormholes with negative and zero cosmological constant. The radial tension is negative, i.e., it is a pressure, for wormholes with the cosmological constant obeying $\Lambda_{\rm int}>1/r_{\rm o}^2$. The total radial tension, $\overline{\tau}(r_{\rm o})=\tau(r_{\rm o})+(c^4/8\pi G)\Lambda_{\rm int}$, is always positive, of course.

2. Exterior vacuum solution of the Einstein equations with generic $\Lambda_{\rm ext}$

The spacetime geometry for a vacuum exterior region is simply determined considering a null stress-energy tensor, $T_{\hat{\mu}\hat{\nu}}=0$, i.e., $\rho(r)=\tau(r)=p(r)=0$. Note that $\Lambda_{\rm ext}$ represents the cosmological constant associated with the exterior solution. In the most general case, the exterior radial coordinate $\overline{r}$ should be different from the interior one r. Here we put them equal; both are denoted by r, since it simplifies the junction and it gives interestingly enough results. The Einstein equations then reduce to

$$0=\frac{b'}{r^2}-\Lambda_{\rm ext}, \tag{33}$$

$$0=\frac{b}{r^3}-2\left(1-\frac{b}{r}\right)\frac{\Phi'}{r}-\Lambda_{\rm ext}, \tag{34}$$

$$0=\left(1-\frac{b}{r}\right)\left[\Phi''+(\Phi')^2-\frac{b'r-b}{2r^2(1-b/r)}\Phi'-\frac{b'r-b}{2r^3(1-b/r)}+\frac{\Phi'}{r}\right]+\Lambda_{\rm ext}. \tag{35}$$

The metric quantities should carry a subscript $_{\rm ext}$, but again we have not included it so as to not overload the notation. Solving the system of differential equations of Eqs. (33)–(35), the exterior vacuum solution with a cosmological constant is given by

$$ds^2=-\left(1-\frac{2GM}{c^2r}-\frac{\Lambda_{\rm ext}}{3}r^2\right)c^2dt^2+\frac{dr^2}{\left(1-\frac{2GM}{c^2r}-\frac{\Lambda_{\rm ext}}{3}r^2\right)}+r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{36}$$

This metric is the unique solution to the vacuum Einstein equations for a static and spherically symmetric spacetime with a generic cosmological constant. The denomination given to it depends on the sign of $\Lambda_{\rm ext}$. The Schwarzschild solution, which is a particular case, is obtained by setting $\Lambda_{\rm ext}=0$. In the presence of a positive cosmological constant, $\Lambda_{\rm ext}>0$, the solution is designated by the Schwarzschild–de Sitter metric. For $\Lambda_{\rm ext}<0$, we have the Schwarzschild–anti de Sitter metric. For $\Lambda\neq 0$, note that this metric is not asymptotically flat as $r\to\infty$; it is either asymptotically de Sitter ($\Lambda_{\rm ext}>0$), or asymptotically anti–de Sitter ($\Lambda_{\rm ext}<0$). However, if $\Lambda_{\rm ext}$ is extremely small, there is a range of the radial coordinate, i.e., $1/\sqrt{\Lambda_{\rm ext}}\gg r\gg GM/c^2$, for which the metric is nearly flat. For values of r below this range, the effect of the mass M dominates, whereas for values above this range, the effect of the cosmological term dominates, as for very large values of r the large-scale curvature of the spacetime must be taken into account.

a. The Schwarzschild spacetime, $\Lambda_{\rm ext}=0$. Equation (36) with $\Lambda_{\rm ext}=0$ is the Schwarzschild solution. The full vacuum solution represents a black hole in an asymptotically flat spacetime. The factor $f(r)=[1-(2GM/c^2r)]$ is zero at

$$r_b=\frac{2GM}{c^2}, \tag{37}$$

the black hole event horizon. Since the wormhole matter will fill the region up to a radius a larger than r_b this radius does not enter into the problem. It is important to have it in mind, since if after construction one finds that $r_b>a$, then the object constructed is a black hole rather than a wormhole.

b. The Schwarzschild–de Sitter spacetime, $\Lambda_{\rm ext}>0$. Equation (36) with $\Lambda_{\rm ext}>0$ represents a black hole in asymptoti-

cally de Sitter space. If $0<9\Lambda_{\rm ext}(GMc^{-2})^2<1$, the factor $f(r)=[1-(2GM/c^2r)-(\Lambda_{\rm ext}/3)r^2]$ is zero at two positive values of r, corresponding to two real positive roots. Defining

$$A=\left(\frac{3c^4}{8\Lambda_{\rm ext}G^2M^2}\right)^{1/3}\sqrt[3]{-1+\sqrt{1-\frac{c^4}{9\Lambda_{\rm ext}G^2M^2}}}, \tag{38}$$

$$B=\left(\frac{3c^4}{8\Lambda_{\rm ext}G^2M^2}\right)^{1/3}\sqrt[3]{-1-\sqrt{1-\frac{c^4}{9\Lambda_{\rm ext}G^2M^2}}}, \tag{39}$$

the solutions are given by

$$r_b=\frac{2GM}{c^2}\left(-\frac{A+B}{2}-\frac{A-B}{2}\sqrt{-3}\right), \tag{40}$$

$$r_c=\frac{2GM}{c^2}(A+B). \tag{41}$$

When $\Lambda_{\rm ext}(GM/c^2)^2\ll 1$, one gets

$$r_b=\frac{2GM}{c^2}\left[1+\frac{4}{3}\Lambda_{\rm ext}\left(\frac{GM}{c^2}\right)^2\right], \tag{42}$$

$$r_c=\sqrt{\frac{3}{\Lambda_{\rm ext}}}\left(1-\frac{GM}{c^2}\sqrt{\frac{\Lambda_{\rm ext}}{3}}\right). \tag{43}$$

The smaller of the values, denoted by $r=r_b$, can be considered as the event horizon of the vacuum black hole solution, but since the wormhole matter will fill the region up to a radius a superior than r_b this radius does not enter into the problem. The larger value, denoted by $r=r_c$, can be regarded as the position of the cosmological event horizon of the de Sitter spacetime. Keeping $\Lambda_{\rm ext}$ constant, but increasing M, $r=r_b$ will increase and $r=r_c$ will decrease. If $9\Lambda_{\rm ext}(GMc^{-2})^2=1$, both horizons coincide and are situated at $r=r_b=r_c=3GM/c^2$. Thus we will consider $9\Lambda_{\rm ext}(GMc^{-2})^2<1$. Particular cases are $\Lambda_{\rm ext}=0$ yielding the Schwarzschild solution, and $M=0$ yielding the de Sitter solution. When $r\to\infty$ the metric tends to the de Sitter spacetime

$$ds^2=-\left(1-\frac{\Lambda_{\rm ext}}{3}r^2\right)c^2dt^2+\frac{dr^2}{\left(1-\frac{\Lambda_{\rm ext}}{3}r^2\right)}+r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{44}$$

For $\Lambda_{\rm ext}\to 0$, the de Sitter metric tends to the Minkowskian spacetime. In the coordinates adopted above, the metric of the de Sitter spacetime will be singular if $r=(3/\Lambda_{\rm ext})^{1/2}$, but this is a mere coordinate singularity signaling the presence of a cosmological event horizon.

c. The Schwarzschild–anti de Sitter spacetime, $\Lambda_{\rm ext}<0$. For the Schwarzschild–anti de Sitter metric, with $\Lambda_{\rm ext}<0$, the equation $f(r)=[1-(2GM/c^2r)+(|\Lambda_{\rm ext}|/3)r^2]=0$ will have only one real root, therefore implying the existence of one horizon. If one defines

$$A=\left(\frac{3c^4}{8|\Lambda_{\rm ext}|G^2M^2}\right)^{1/3}\times\sqrt[3]{1+\sqrt{1+\frac{c^4}{9|\Lambda_{\rm ext}|G^2M^2}}}, \tag{45}$$

$$B=\left(\frac{3c^4}{8|\Lambda_{\rm ext}|G^2M^2}\right)^{1/3}\times\sqrt[3]{1-\sqrt{1+\frac{c^4}{9|\Lambda_{\rm ext}|G^2M^2}}}, \tag{46}$$

then the solution is

$$r_b=\frac{2GM}{c^2}(A+B). \tag{47}$$

For $|\Lambda_{\rm ext}|(GM/c^2)^2\ll 1$ one obtains

$$r_b=\frac{2GM}{c^2}\left[1-\frac{4}{3}|\Lambda_{\rm ext}|\left(\frac{GM}{c^2}\right)^2\right]. \tag{48}$$

Once again this event horizon is avoided by filling the space with exotic matter from the throat at r_o up to the mouth at a, where $a>r_b$ in order that the wormhole is not a black hole. If $\Lambda_{\rm ext}=0$, the metric is reduced to the Schwarzschild solution. If $r\to\infty$ the metric tends to the anti–de Sitter solution

$$ds^2=-\left(1+\frac{|\Lambda_{\rm ext}|}{3}r^2\right)c^2dt^2+\frac{dr^2}{\left(1+\frac{|\Lambda_{\rm ext}|}{3}r^2\right)}+r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{49}$$

For $|\Lambda_{\rm ext}|\to 0$, the anti–de Sitter metric tends to the Minkowskian spacetime.

3. Junction conditions in wormholes with generic $\Lambda_{\rm ext}$

To match the interior to the exterior, one needs to apply the junction conditions that follow from the theory of general relativity. One of the conditions imposes the continuity of the metric components, $g_{\mu\nu}$, across a surface, S, i.e., $g_{\mu\nu({\rm int})}|_S=g_{\mu\nu({\rm ext})}|_S$. This condition is not sufficient to join different spacetimes. One formalism of matching, that leads to no errors in the calculation, uses the extrinsic curvature of S (or second fundamental form of the surface S, the first fundamental form being the metric on S) see, e.g., [82]. However, for spacetimes with a good deal of symmetry, such as spheri-

cal symmetry, one can use directly the field equations to make the match, see e.g., [83] (see also Taub [84]). We follow this latter approach. Indeed, due to the high symmetries of the solution, we can use the Einstein equations, Eqs. (16)–(18), to determine the energy density and stresses of the surface S necessary to have a match between the interior and exterior. If there are no surface stress-energy terms at the surface S, the junction is called a boundary surface. If, on the other hand, surface stress-energy terms are present, the junction is called a thin-shell.

a. Matching of the metric. As was mentioned above, the unique vacuum static and spherically symmetric solution, in the presence of a nonvanishing cosmological constant, is given by Eq. (36). A wormhole with finite dimensions, in which the matter distribution extends from the throat, $r=r_0$, to a finite distance $r=a$, obeys the condition that the metric is continuous. Due to the spherical symmetry the components $g_{\theta\theta}$ and $g_{\phi\phi}$ are already continuous, and so one is left with imposing the continuity of g_{tt} and g_{rr},

$$g_{tt(\mathrm{int})}=g_{tt(\mathrm{ext})}, \tag{50}$$

$$g_{rr(\mathrm{int})}=g_{rr(\mathrm{ext})}, \tag{51}$$

at $r=a$, with $g_{tt(\mathrm{int})}$ and $g_{rr(\mathrm{int})}$ being the metric components for the interior region at $r=a$, and $g_{tt(\mathrm{ext})}$ and $g_{rr(\mathrm{ext})}$ the exterior metric components for the vacuum solution at $r=a$. For the sake of consistency in the notation this could have been done in the orthonormal frame with the hat quantities $g_{\hat{\mu}\hat{\nu}}$, but in this case it is more direct to do with coordinate frame quantities. One can start the analysis by considering two general solutions of Eq. (1), an interior solution and an exterior solution matched at a surface, S. The continuity of the metric then gives generically $\Phi_{\mathrm{int}}(a)=\Phi_{\mathrm{ext}}(a)$ and $b_{\mathrm{int}}(a)=b_{\mathrm{ext}}(a)$. If one now uses, Eqs. (1), (36), (50) and (51), one finds then $e^{2\Phi(a)}=[1-2GM/(c^2a)-\Lambda_{\mathrm{ext}}a^2/3]$ and $[1-b(a)/a]=[1-2GM/(c^2a)-\Lambda_{\mathrm{ext}}a^2/3]$ which can be simplified to

$$\Phi(a)=\frac{1}{2}\ln\left(1-\frac{2GM}{c^2a}-\frac{\Lambda_{\mathrm{ext}}}{3}a^2\right), \tag{52}$$

$$b(a)=\frac{2GM}{c^2}+\frac{\Lambda_{\mathrm{ext}}}{3}a^3. \tag{53}$$

From Eq. (53), one deduces that the mass of the wormhole is given by

$$M=\frac{c^2}{2G}\left(b(a)-\frac{\Lambda_{\mathrm{ext}}}{3}a^3\right). \tag{54}$$

b. Matching of the equations I: The surface pressure. We are going to consider the case where static interior observers measure zero tidal forces, i.e., $\Phi_{\mathrm{int}}=\mathrm{const}$, $\Phi'_{\mathrm{int}}=0$. This is, of course, a particular choice which simplifies the analysis. As we have seen, the metric is continuous through the surface S. However, their first and second derivatives might not be. Since the metric is static and spherically symmetric the only derivatives that one needs to worry about are radial. Now, second derivatives in the metric are related to the Einstein tensor $G_{\mu\nu}$, or since we are working with hat quantities, to $G_{\hat{\mu}\hat{\nu}}$. But $G_{\hat{\mu}\hat{\nu}}$ is proportional to the stress-energy tensor $T_{\hat{\mu}\hat{\nu}}$. Thus, something in the stress-energy tensor has to reflect this discontinuity. Indeed, at the boundary S, $T_{\hat{\mu}\hat{\nu}}$ is proportional to a Dirac δ function, and we can write $T_{\hat{\mu}\hat{\nu}}=t_{\hat{\mu}\hat{\nu}}\delta(\hat{r}-\hat{a})$, where $\hat{r}=\sqrt{g_{rr}}r$ means the proper distance through the thin-shell. To find $t_{\hat{\mu}\hat{\nu}}$ one then uses $\int_-^+ G_{\hat{\mu}\hat{\nu}}d\hat{r}=(8\pi G/c^4)\int_-^+ t_{\hat{\mu}\hat{\nu}}\delta(\hat{r}-\hat{a})d\hat{r}$, where $\int_-^+$ means an infinitesimal integral through the shell. Using the property of the δ function $\delta(f(x))=[1/|f'(x)|]\delta(x)$, and $\int_-^+ g(x)\delta(x-x_0)=g(x_0)$, one finds

$$t_{\hat{\mu}\hat{\nu}}=\frac{c^4}{8\pi G}\int_-^+ G_{\hat{\mu}\hat{\nu}}d\hat{r}. \tag{55}$$

Since the shell is infinitesimally thin in the radial direction there is no radial pressure, thus we are left with a surface energy term Σ, and a surface tangential pressure $\mathcal{P}$.

First we calculate the surface energy density Σ. From Eq. (6) we see that $G_{\hat{t}\hat{t}}$ only depends on first derivatives of the metric, so that when integrated through the shell it will give metric functions only, which by definition are continuous. Thus, since the integral gives the value of the metric on the exterior side (b^+, say) minus the value of the metric on the interior side (b^-), it gives zero, and one finds

$$\Sigma=0. \tag{56}$$

Now we find the surface tangential pressure $\mathcal{P}$. From Eq. (8) we see that $G_{\hat{\theta}\hat{\theta}}$ has an important term $[1-(b/r)]\Phi''$. The other terms depend at most on the first derivative and as before do no contribute to the integral. Thus, in this case Eq. (55) gives $8\pi G/c^2\mathcal{P}=\sqrt{1-b(a)/a}\,\Phi'^+_-$. Now, $\Phi'_-=0$ by assumption, and $\Phi'^+=[GM/(c^2a^2)-\Lambda_{\mathrm{ext}}a/3]/[1-b(a)/a]$. Thus,

$$\mathcal{P}=\frac{c^4}{8\pi Ga}\,\frac{GM/(c^2a^2)-\Lambda_{\mathrm{ext}}a/3}{\sqrt{1-b(a)/a}},$$

or more explicitly,

$$\mathcal{P}=\frac{c^4}{8\pi Ga}\,\frac{\dfrac{GM}{c^2a}-\dfrac{\Lambda_{\mathrm{ext}}}{3}a^2}{\sqrt{1-\dfrac{2GM}{c^2a}-\dfrac{\Lambda_{\mathrm{ext}}}{3}a^2}}. \tag{57}$$

$\mathcal{P}$ is always positive for the Schwarzschild and the Schwarzschild–anti de Sitter spacetime, i.e., $\Lambda_{\mathrm{ext}}\leqslant 0$. The Schwarzschild–de Sitter spacetime, $\Lambda_{\mathrm{ext}}>0$, has to be analyzed more carefully. In Fig. 1, we plot in a graph $9\Lambda_{\mathrm{ext}}(GM/c^2)^2\times 2GM/(c^2a)$ the regions where $\mathcal{P}$ is negative, zero or positive. For high $\Lambda_{\mathrm{ext}}(GMc^{-2})^2$ (either Λ_{ext} big or M big) and at low $2GM/(c^2a)$ (either M small or a big), one needs a surface tension to support the structure. In the other case one needs a surface pressure. This is expected

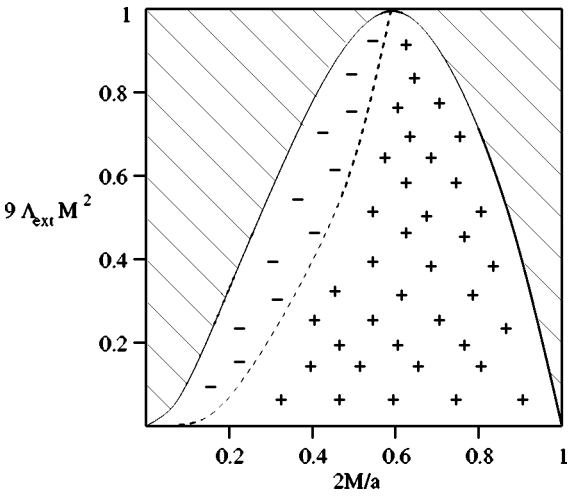

FIG. 1. The regions where $\mathcal{P}$ is negative and positive in a plot of $9\Lambda_{\rm ext}(GM/c^2)^2$ as a function of the inverse of the relative size of the wormhole, i.e., $2GM/c^2a$ (in the figure we have used geometrical units $G=1=c$) are given. Inside the solid line is the region of solutions. To the left of the dashed line $\mathcal{P}$ is a negative pressure, i.e., a tension, to the right $\mathcal{P}$ is a positive quantity, i.e., is a pressure, see text for more details.

in the sense that for a positive $\Lambda_{\rm ext}$ one has an expanding external de Sitter spacetime. If a is big [and so $2GM/(c^2a)$ small], the wormhole boundary is participating somehow in the expansion, so one needs a tension to hold it. For small a ($2GM/c^2a$ big), the gravity wins over the expansion and so one needs a pressure to hold against collapse, a particular case being the Schwarzschild case $\Lambda_{\rm ext}=0$.

One can have a term of comparison for the surface tangential pressure $\mathcal{P}$ at the thin-shell. Assuming that the thin-shell has a width of approximately Δr, one can consider a volumetric tangential pressure, orthogonal to the radial coordinate, given by

$$\tilde{P}=\frac{\mathcal{P}}{\Delta r}. \tag{58}$$

Taking into account Eq. (31), we see that the tangential pressure at the mouth, with $\Phi'_{\rm int}=0$, is given by $\overline{p(a)}=(c^4/16\pi Ga^3)[b(a)-b'(a)a]$. Estimates of $\tilde{P}$ may be given in terms of $p(a)$, by defining the following ratio:

$$\frac{\tilde{P}}{\overline{p(a)}}=2\frac{a}{\Delta r}\frac{1}{\dfrac{b}{a}-b'}\frac{\dfrac{GM}{c^2a}-\dfrac{\Lambda_{\rm ext}}{3}a^2}{\sqrt{1-\dfrac{2GM}{c^2}-\dfrac{\Lambda_{\rm ext}}{3}a^3}}. \tag{59}$$

It is also interesting to find the ratio to $p(r_{\rm o})$, the maximum pressure, given by

$$\frac{\tilde{P}}{p(r_{\rm o})}=2\frac{r_{\rm o}^2}{a\Delta r}\frac{1}{1-b'(r_{\rm o})}\frac{\dfrac{GM}{c^2a}-\dfrac{\Lambda_{\rm ext}}{3}a^2}{\sqrt{1-\dfrac{2GM}{c^2a}-\dfrac{\Lambda_{\rm ext}}{3}a^2}}. \tag{60}$$

One may find numerical estimates, considering various choices of the shape function, $b(r)$, which will be done while considering specific solutions of traversable wormholes.

c. Matching of the equations II: The radial pressure. To construct specific solutions of wormholes with generic cosmological constant, one needs to know how the radial tension behaves across the junction boundary, *S*. The analysis is simplified if we consider two general solutions of Eq. (1), an interior solution and an exterior solution matched at a surface, *S*. The radial component of the Einstein equations, Eq. (17), provides

$$\frac{b_{\rm int}}{r^3}=\frac{8\pi G}{c^4}\tau_{\rm int}(r)+\Lambda_{\rm int}+2\left(1-\frac{b_{\rm int}}{r}\right)\frac{\Phi'_{\rm int}}{r}, \tag{61}$$

$$\frac{b_{\rm ext}}{r^3}=\frac{8\pi G}{c^4}\tau_{\rm ext}(r)+\Lambda_{\rm ext}+2\left(1-\frac{b_{\rm ext}}{r}\right)\frac{\Phi'_{\rm ext}}{r}. \tag{62}$$

Taking into account the continuity of the metric at the junction boundary one has obtained $\Phi_{\rm int}(a)=\Phi_{\rm ext}(a)$ and $b_{\rm int}(a)=b_{\rm ext}(a)$. For simplicity, we are considering $\Phi'_{\rm int}(a)=0$. Using again the relation

$$\Phi'_{\rm ext}(r)=\left(\frac{GM}{c^2r^2}-\frac{\Lambda_{\rm ext}}{3}r\right)\Big/\left(1-\frac{2GM}{c^2r}-\frac{\Lambda_{\rm ext}}{3}r^2\right),$$

and taking into account Eq. (57), we verify that Eqs. (61), (62) provide us with an equation which governs the behavior of the radial tension at the boundary, namely,

$$\tau_{\rm int}(a)+\frac{c^4}{8\pi G}\Lambda_{\rm int}=\tau_{\rm ext}(a)+\frac{c^4}{8\pi G}\Lambda_{\rm ext}+\frac{2}{a}\mathcal{P}e^{\Phi(a)}, \tag{63}$$

where we have put $e^{\Phi(a)}=\sqrt{1-2GM/(c^2a)-\Lambda_{\rm ext}a^2/3}$. Equation (63), although not new in its most generic form [2], is a beautiful equation that relates the radial tension at the surface with the tangential pressure of the thin-shell. A particularly interesting case is when $\mathcal{P}=0$. In this situation $M=\Lambda_{\rm ext}c^2a^3/(3G)$, and $\Phi'_{\rm ext}(a)=0$. Since by our construction $\Phi'_{\rm int}(a)=0$, Φ' is continuous across the surface and the solution is reduced to a boundary surface. From Eq. (53) one finds that the shape function at the junction is given by $b(a)=\Lambda_{\rm ext}a^3$. Thus, Eq. (63) simplifies to, $\tau_{\rm int}(a)+(c^4/8\pi G)\Lambda_{\rm int}=\tau_{\rm ext}(a)+(c^4/8\pi G)\Lambda_{\rm ext}$. If one considers a matching of an interior solution of a wormhole with generic Λ, given by Eq. (1), to an exterior Schwarzschild solution, with $\tau_{\rm ext}=0$ and $\Lambda_{\rm ext}=0$, we simply have the condi-

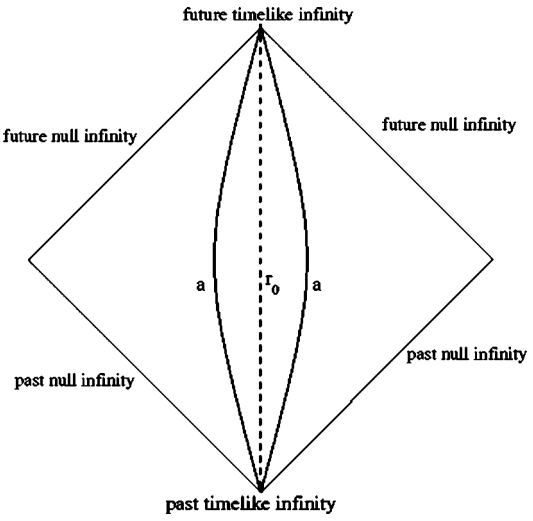

FIG. 2. The spacetime diagram for the wormhole with $\Lambda_{ext}=0$, i.e., a wormhole in an asymptotically Minkowski spacetime, represented by two copies of the Minkowski diagram joined at the throat.

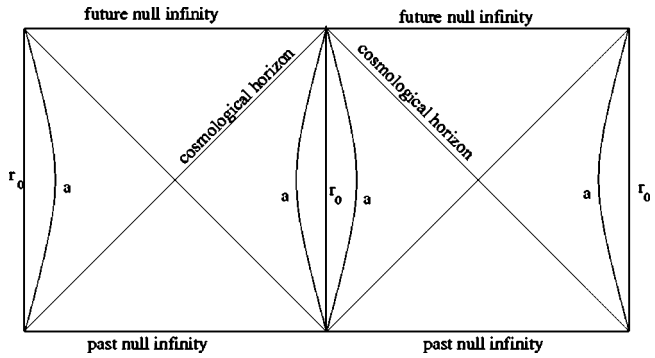

FIG. 3. The spacetime diagram for the wormhole with $\Lambda_{ext}>0$, i.e., a wormhole in an asymptotically de Sitter spacetime, with an infinite number of copies (only two are represented).

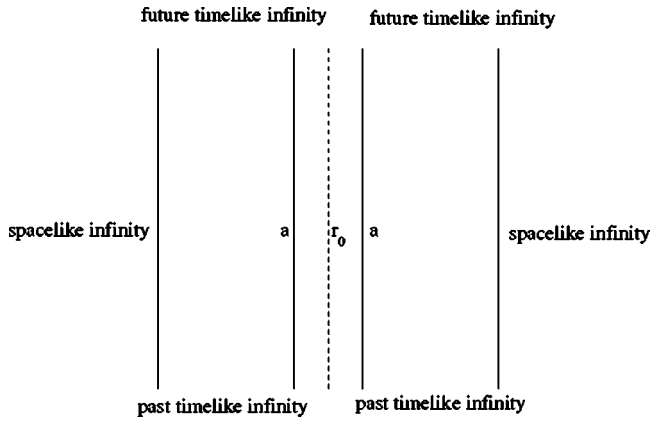

FIG. 4. The spacetime diagram for the wormhole with $\Lambda_{ext}<0$, i.e., a wormhole in an asymptotically anti–de Sitter spacetime, represented by two copies of the anti–de Sitter diagram joined at the throat.

tion that $\tau_{int}(a)+(c^4/8\pi G)\Lambda_{int}=0$ at the boundary surface. Matching an interior solution to an exterior Schwarzschild–de Sitter or Schwarzschild–anti–de Sitter solution, with $\tau_{ext}=0$ and $\Lambda_{ext}\neq 0$, we have the relationship $\tau_{int}(a)+(c^4/8\pi G)\Lambda_{int}=(c^4/8\pi G)\Lambda_{ext}$ at the boundary surface. These solutions will be analyzed in the next sections.

D. Spacetime diagrams

We now draw the spacetime diagrams, i.e., the Carter-Penrose diagrams, corresponding to wormholes in spacetimes with $\Lambda_{ext}=0$, $\Lambda_{ext}>0$, and $\Lambda_{ext}<0$. They are easy to sketch once one knows the corresponding diagrams for the solution with no wormhole, i.e., the Carter-Penrose diagrams for Minkowski spacetime, de Sitter spacetime and anti–de Sitter spacetime [8], respectively. Each point in the diagram represents a sphere. Since the wormhole creates an extra asymptotic region one has to duplicate the original diagram through the throat. In Figs. 2–4 the diagrams for a wormhole in an asymptotically flat spacetime $\Lambda_{ext}=0$, in an asymptotically de Sitter spacetime $\Lambda_{ext}>0$, and in an asymptotically anti–de Sitter spacetime $\Lambda_{ext}<0$, respectively, are drawn. Note the duplication of the asymptotic regions.

III. SPECIFIC CONSTRUCTION OF WORMHOLES WITH GENERIC Λ

We will give some simple examples of traversable wormholes similar to those constructed in [1]. The difference from the wormholes in that work is that the wormholes here in general have an infinitesimal thin-shell with a tangential pressure $\mathcal{P}\neq 0$, and the exterior spacetime has a cosmological constant. We discuss briefly the three cases $\Lambda_{ext}=0$, $\Lambda_{ext}>0$, and $\Lambda_{ext}<0$.

A. Specific solutions of traversable wormholes with $\Lambda_{ext}=0$ (asymptotically flat wormholes)

1. Matching to an exterior Schwarzschild solution, with $\mathcal{P}=0$

Here we consider a matching of an interior solution with an exterior Schwarzschild solution ($\tau_{ext}=0$ and $\Lambda_{ext}=0$), and with the junction having zero tangential pressure, $\mathcal{P}=0$. From Eq. (63) one has at the junction

$$\tau_{int}(a)+\frac{c^4}{8\pi G}\Lambda_{int}=0. \tag{64}$$

Then, from Eq. (30) (with $\Phi'=0$) one gets

$$0=\frac{c^4}{8\pi G}\frac{b(a)}{a^3}. \tag{65}$$

Since $b\neq 0$, Eq. (65) is only satisfied if $a\to\infty$. This is one of the cases considered by Morris and Thorne [1], in which the wormhole's material extends from the throat all the way to infinity.

2. Matching to an exterior Schwarzschild solution, with $\mathcal{P}\neq 0$

Matching the interior solution to an exterior Schwarzschild solution ($\tau_{\rm ext}=0$ and $\Lambda_{\rm ext}=0$) but considering $\mathcal{P}\neq 0$ provides some interesting results. The behavior of the radial tension at the junction is given by Eq. (63) and taking into account Eq. (53) one finds that the shape function at the junction simply reduces to $b(a)=2GM/c^2$. We will next consider various choices for the shape function, $b(r)$, which will give different wormhole solutions.

a. $b(r)=(r_{\rm o}r)^{1/2}$. Consider the following functions:

$$\Phi(r)=\Phi_0, \tag{66}$$

$$b(r)=(r_{\rm o}r)^{1/2}, \tag{67}$$

where $r_{\rm o}$ is the throat radius as before. Using the Einstein equations, Eqs. (29)–(31), we have

$$\overline{\rho}(r)\equiv\rho(r)+\frac{c^2}{8\pi G}\Lambda_{\rm int}=\frac{c^2}{8\pi G}\frac{r_{\rm o}^{1/2}}{2r^{5/2}}, \tag{68}$$

$$\overline{\tau}(r)\equiv\tau(r)+\frac{c^2}{8\pi G}\Lambda_{\rm int}=\frac{c^4}{8\pi G}\frac{r_{\rm o}^{1/2}}{r^{5/2}}, \tag{69}$$

$$\overline{p}(r)\equiv p(r)+\frac{c^2}{8\pi G}\Lambda_{\rm int}=\frac{c^4}{8\pi G}\frac{r_{\rm o}^{1/2}}{4r^{5/2}}. \tag{70}$$

To find an estimate of the surface pressure at the thin-shell, one has $\tilde{P}/p(r_{\rm o})=4r_{\rm o}^2GM/\Delta rc^2a^2[1-(2GM/c^2a)]^{-1/2}$ [see Eq. (60)].

From $b(a)=2GM/c^2$ and $b(a)=(r_{\rm o}a)^{1/2}$, one finds that the matching occurs at

$$a=\frac{(2GM/c^2)^2}{r_{\rm o}}. \tag{71}$$

Now in order that the wormhole is not a black hole one has to impose $a>2GM/c^2$. Then, from Eq. (71) one finds $r_{\rm o}<2GM/c^2$. From Eq. (71), we also extract the mass of the wormhole, given by $M=c^2(r_{\rm o}a)^{1/2}/(2G)$.

The interior metric, $r_{\rm o}\leqslant r\leqslant a$, is determined recalling that $e^{2\Phi(a)}=[1-\sqrt{r_{\rm o}/a}]$. It is given by

$$ds^2=-\left(1-\sqrt{\frac{r_{\rm o}}{a}}\right)c^2dt^2+\frac{dr^2}{\left(1-\sqrt{\frac{r_{\rm o}}{r}}\right)}+r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{72}$$

The exterior metric, $a\leqslant r<\infty$, is given by

$$ds^2=-\left(1-\frac{\sqrt{r_{\rm o}a}}{r}\right)c^2dt^2+\frac{dr^2}{\left(1-\frac{\sqrt{r_{\rm o}a}}{r}\right)}+r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{73}$$

The final metric of the whole spacetime is given by Eqs. (72), (73), which are joined smoothly, as we have carefully worked out.

It is also interesting to briefly consider the traversability conditions that the absurdly advanced civilization might require to cross the wormhole from one mouth to the other and back [1]. One finds that for an observer traversing the wormhole with a velocity $v=0.01c$, the wormhole has a throat radius given by $r_{\rm o}\geqslant 500$ km. One can choose $r_{\rm o}=500$ km, and assume that the traversal time done by the spaceship is approximately one year. Then, one finds that the matter distribution extends from $r_{\rm o}$ to $a=4.74\times10^{13}$ m$\approx 5\times10^{-3}$ light years, with a being the size of the wormhole. It is supposed that the space stations are parked there. One also finds that the wormhole mass is $M=3.3\times10^{36}$ kg, six orders of magnitude superior to the Sun's mass. One may also find an estimate for Eq. (60), giving $\tilde{P}/p(r_{\rm o})\approx10^{-4}$.

One may choose other parameters, for instance, so that the wormhole mass is of the same order of the Sun's mass. Considering a traversal with a velocity $v=5.4\times10^3$ m/s, we may choose that the wormhole throat is given by $r_{\rm o}=9\times10^2$ m. If we consider an extremely fast trip, where the traversal time is given by $\Delta\tau_{\rm traveler}=3.7$ s, the matter distribution extends from $r_{\rm o}$ to $a=10^4$ m. In this case the mass of the wormhole is given by $M\approx2\times10^{30}$ kg, which is the Sun's mass. From Eq. (60) an estimate to $\tilde{P}$ is $\tilde{P}/p(r_{\rm o})\approx5.7\times10^3$. We have an extremely large surface pressure. As the wormhole mass is decreased, one sees that a larger tangential surface pressure is needed to support the structure.

b. $b(r)=r_{\rm o}^2/r$. Consider now,

$$\Phi(r)=\Phi_0, \tag{74}$$

$$b(r)=r_{\rm o}^2/r. \tag{75}$$

One can use again the Einstein equations (29)–(31) to find the properties of this wormhole. We will not do it here. The interior solution is the same as the one found by Ellis [18]. The properties are commented on in [1,22]. Harris showed further that it is a solution of the Einstein equations with a stress-energy tensor of a peculiar massless scalar field [23]. From the traversability conditions one finds that this type of wormhole can have much lower masses than the previous type. One can find wormholes about the Earth's mass.

B. Specific solutions of traversable wormholes with $\Lambda_{\rm ext}>0$ (asymptotically de Sitter wormholes)

1. Matching to an exterior Schwarzschild–de Sitter solution, with $\mathcal{P}=0$

In this section we will be interested in a matching of an interior solution with an exterior Schwarzschild–de Sitter solution, $\tau_{\rm ext}=0$ and $\Lambda_{\rm ext}>0$, at a boundary surface, $\mathcal{P}=0$. We verify from Eq. (63) that the following condition holds:

$$\tau_{\rm int}(a)+\frac{c^4}{8\pi G}\Lambda_{\rm int}=\frac{c^4}{8\pi G}\Lambda_{\rm ext}, \tag{76}$$

at the surface boundary. Considering Eq. (30), we have

$$b(a)=\Lambda_{\rm ext}a^3. \tag{77}$$

Substituting this value in Eq. (54), one obtains the mass of the wormhole, given by

$$M=\frac{c^2}{3G}\Lambda_{\rm ext}a^3. \tag{78}$$

We shall next consider identical shape functions as in the above section.

a. $b(r)=(r_{\rm o}r)^{1/2}$. Consider the following functions:

$$\Phi(r)=\Phi_0, \tag{79}$$

$$b(r)=(r_{\rm o}r)^{1/2}. \tag{80}$$

From $b(a)=\Lambda_{\rm ext}a^3$ and $b(a)=(r_{\rm o}a)^{1/2}$, one sees that the matching occurs at

$$a\equiv\frac{r_{\rm o}^{1/5}}{\Lambda_{\rm ext}^{2/5}}. \tag{81}$$

The mass can then be expressed as $M=c^2(r_{\rm o}a)^{1/2}/(3G)$. It can be shown that the interior solution, $r_{\rm o}\leqslant r\leqslant a$, is identical to Eq. (72), i.e.,

$$ds^2=-\left(1-\sqrt{\frac{r_{\rm o}}{a}}\right)c^2dt^2+\frac{dr^2}{\left(1-\sqrt{\frac{r_{\rm o}}{r}}\right)}+r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{82}$$

The exterior solution, $a\leqslant r<\infty$, is given by the following metric:

$$ds^2=-\left(1-\frac{2(r_{\rm o}a)^{1/2}}{3r}-\frac{r_{\rm o}^{1/2}r^2}{3a^{5/2}}\right)c^2dt^2+\frac{dr^2}{\left(1-\frac{2(r_{\rm o}a)^{1/2}}{3r}-\frac{r_{\rm o}^{1/2}r^2}{3a^{5/2}}\right)}+r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{83}$$

The spacetime of the final solution is given by the metrics, Eqs. (82), (83), which have been smoothly joined at a.

The additional parameter now is the cosmological constant, $\Lambda_{\rm ext}$, given by $\Lambda_{\rm ext}=(r_{\rm o}/a^5)^{1/2}$. For instance, consider a traversal velocity $v=0.01c$, so that $r_{\rm o}=5\times10^5$ m. If the observer traverses through the wormhole comfortably during a year, $\Delta\tau_{\rm traveler}\approx3.16\times10^7$ s, and $a=4.74\times10^{13}$ m. The mass of the wormhole is $M\approx2.2\times10^{36}$ kg and the cosmological constant has the value $\Lambda_{\rm ext}=4.6\times10^{-32}\ {\rm m}^{-2}$. The cosmological event horizon is then situated at $r_c=8.1\times10^{15}\ {\rm m}\approx200a$.

b. $b(r)=r_{\rm o}^2/r$. Consider now the functions

$$\Phi(r)=\Phi_0, \tag{84}$$

$$b(r)=r_{\rm o}^2/r. \tag{85}$$

One can use the same procedure to find the properties of this wormhole. In this case one can find a wormhole with the mass of an asteroid ($M\simeq1.5\times10^{19}$ Kg) for a cosmological constant with a value equal to its present value ($\Lambda_{\rm ext}\simeq10^{-54}\ {\rm m}^{-2}$).

2. Matching to an exterior Schwarzschild–de Sitter solution, with $\mathcal{P}\neq0$

One can also match the interior solution with an exterior Schwarzschild–de Sitter solution ($\tau_{\rm ext}=0$ and $\Lambda_{\rm ext}>0$) in the presence of a thin-shell, $\mathcal{P}\neq0$. From Eq. (63), we have the behavior of the radial tension at the thin-shell, given by

$$\tau_{\rm int}(a)+\frac{c^4}{8\pi G}\Lambda_{\rm int}=\frac{c^4}{8\pi G}\Lambda_{\rm ext}+\frac{2}{a}\mathcal{P}e^{\Phi(a)}. \tag{86}$$

The shape function at the junction is given by Eq. (53). From Eq. (54), one verifies that the mass of the wormhole is zero when $b(a)=\Lambda_{\rm ext}a^3/3$, is positive when $b(a)>\Lambda_{\rm ext}a^3/3$, and is negative when $b(a)<\Lambda_{\rm ext}a^3/3$. One can perform a similar analysis as done for the previous examples.

C. Specific solutions of traversable wormholes with $\Lambda_{\rm ext}<0$ (asymptotically anti–de Sitter wormholes)

1. Matching to an exterior Schwarzschild–anti–de Sitter solution, with $\mathcal{P}=0$

From Eq. (63), matching an interior solution with an exterior given by the Schwarzschild–anti–de Sitter solution ($\tau_{\rm ext}=0$ and $\Lambda_{\rm ext}<0$), at a boundary surface $\mathcal{P}=0$, yields

$$\tau_{\rm int}(a)+\frac{c^4}{8\pi G}\Lambda_{\rm int}=-\frac{c^4}{8\pi G}|\Lambda_{\rm ext}|, \tag{87}$$

at the surface boundary. Considering Eq. (30), we have

$$b(a)=-|\Lambda_{\rm ext}|a^3. \tag{88}$$

From Eq. (23), we concluded that the shape function has to be positive to guarantee that the factor $\sqrt{r/b-1}$ is real. Therefore, for the anti–de Sitter exterior, i.e., $\Lambda_{\rm ext}<0$, with $\mathcal{P}=0$ there is no solution. This problem may be overcome by considering a matching to an exterior anti–de Sitter solution with a thin-shell, i.e., $\mathcal{P}\neq0$.

2. Matching to an exterior Schwarzschild–anti–de Sitter solution, with $\mathcal{P}\neq0$

From Eq. (53), one finds that $b(a)$ is positive if

$$\frac{2GM}{c^2}\geqslant\frac{|\Lambda_{\rm ext}|}{3}a^3. \tag{89}$$

Then one can construct easily wormholes in anti–de Sitter spacetime, and again perform a similar analysis as done for the previous examples.

IV. CONCLUSIONS

We have considered Morris-Thorne wormholes, i.e., static and spherically symmetric traversable wormholes, in the presence of a nonvanishing cosmological constant. Matching the interior solution with a vacuum exterior solution, we have deduced an equation for the tangential surface pressure, and another one which governs the behavior of the radial tension at the boundary.

Specific solutions with various choices of the shape function were briefly presented. Through the traversability conditions, we have obtained estimates for the matching boundary, a, the mass of the wormhole M, and the tangential surface pressure $\mathcal{P}$, by imposing values for the traversal velocity and the traversal time.

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